2.9 The Cutting Plane Algorithm: Chv*á*tal-Gomory Cuts

We said in last section that by adding one or more constraints to the LP relaxation of the ILP and then solve the LP relaxation via ordinary solution method, we can possibly obtain a feasible integer solution. What is attractive now is to generate such valid inequalities systematically.

From **Polyhedral Theory** we have seen that *any* IP *problem can be reduced to a* LP, which means it is always possible to find a (rational) linear system

$$conv(S) = \{x \in R^n : \bar{A}x \leq \bar{b}\};$$

Such that

$z = \{\max c^T x : Ax \leq b,\ x\ integer\}$; Equals $\bar{z} = \{\max c^T x :\ x\ \in conv(S)\}$;

Remarks

This might be confusing since we know that integer linear programming is **NP-hard** while linear programming is **polynomial**. However, our reduction merely says that there *exists* such a linear system so it may be *much larger* than the original system $Ax \leq b$.
In fact, size of the reformulation $\bar{A}x \leq \bar{b}$ often **grows exponentially** as a function of the natural size of the original IP.

Despite exponentially growth in size, one may still optimize fast over a polyhedron. Now a basic question is how to find these inequalities defining a convex hull, both in theory and practice.

Clearly, each inequality in $Ax \leq b$ is valid for *conv*(*S*). In this section we discuss methods for finding *additional* **valid inequalities** for *conv*(*S*).

Geometrically, as *conv*(*S*) is spanned by the integer points in IP, what we need is some procedure for ”pushing” a hyperplane defined by an inequality in the system $Ax \leq b$ as far as possible towards *conv*(*S*). Two obvious observations with respect to valid inequalities are:

1. Multiplying a valid inequality by a positive number gives another (equivalent) valid inequality.
2. The sum of valid inequalities is again a valid inequality. We have seen some practices of point 1 in **bound tightening** procedures where we move a hyperplane parallel towards the feasible region. Now we see some utilities of

point 2. We generate new valid inequality by <u>enumerating over linear combination of existing inequalities</u>

Given a set of valid inequalities $a_j^T x \leq b_j, j = 1, \dots, m$ and $\lambda_j \geq 0, j = 1, \dots, m$ the inequality

$$\sum_{j=1}^{m} \lambda_j \, (a_j^T x) \leq \sum_{j=1}^{m} \lambda_j \, b_j$$

Is also valid. As $x \geq 0$, we have

$$\sum_{j=1}^{m} \lfloor \lambda_j \, a_j^T \rfloor x \leq \sum_{j=1}^{m} \lambda_j \, b_j$$

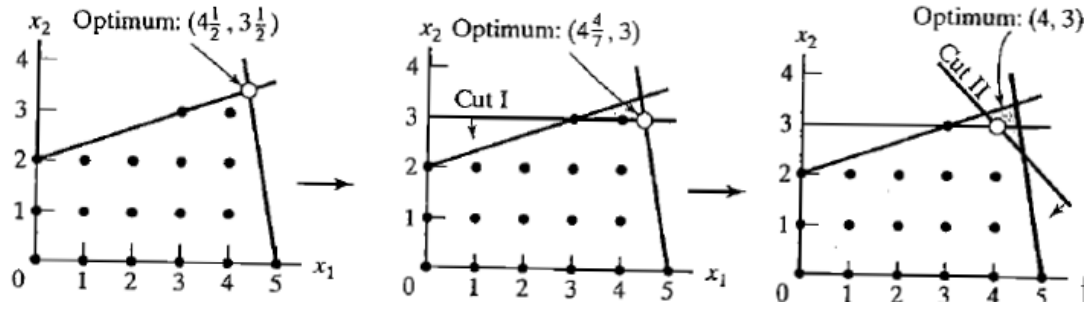
Now the l.h.s. of above inequality can only be integral with integer *x* values, therefore applying the bound-tightening procedure we have

$$\sum_{j=1}^{m} \lfloor \lambda_j a_j^T \rfloor x \leq \lfloor \sum_{j=1}^{m} \lambda_j b_j \rfloor.$$

This is called the **Chv*á*tal-Gomory procedure**. Although it is simple, the **Chv*á*tal-Gomory procedure** is enough to generate ALL valid inequalities for an integer program as it can be applied repeatedly on *existing and newly generated* inequalities.

Example:

$$-x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 35$$
$$x_1, x_2 \geq 0 \; and \; integer$$



However, this characteristic of infinite loop makes the Chv*á*tal-Gomory procedure not a practical method in most applications.

More on valid inequality
Two valid inequalities $\pi x \leq \pi_0 \; and \; \gamma x \leq \gamma_0$ are equivalent if

$$(\gamma, \gamma_0) = \lambda(\pi, \pi_0).$$

When they are not equivalent but:

$$\exists \mu > 0 \; such \; that \;\; \gamma \geq \mu\pi \; and \;\; \gamma_0 \leq \mu \, \pi_0 \, ,$$

Then $\pi x \leq \pi_0$ is dominated by $\gamma x \leq \gamma_0$.

Example: Prove that inequality $x_1 + x_2 \leq 6$ dominates $3x_1 + 4x_2 \leq 24$.

SOLUTION
Consider $\mu = 1/4$ the following two inequalities are true:

$$(1,1) \geq \frac{1}{4} * (3,4)$$
$$6 \leq \frac{1}{4} * 24$$

Remarks
• A maximal valid inequality is one which is not dominated by any other valid inequalities.
• Any maximal valid inequality defines a non-empty face of *conv*(*S*).
• The set of maximal valid inequalities contains all of the facets defining inequalities for *conv*(*S*).

2.10 The Cutting Plane Algorithm: Gomory's Fractional Cuts

A valid inequality for *S* which does not exclude any integer solutions can be added to the LP relaxation without affecting the final solution. Such linear constraint is called a cutting plane (or a cut).

A systematic way of generating cutting planes is due to ***Ralph E. Gomory*** and is called **Gomory's fractional cutting plane algorithm**. This method, instead of generating new valid inequalities by enumerating over linear combinations of existing inequalities, trying to generate cuts from the optimal simplex tableau for the LP relaxation of an IP.

Suppose the LP relaxation is solved to optimality using the simplex algorithm. From the final tableau, a typical row corresponds to the equation:

$$x_{B(i)} + \sum_{j \notin B} y_{ij} x_j = y_{i0}\,, \qquad i = 0, 1, \dots, m \qquad (1)$$

Recall that simplex algorithm only works for nonnegative variables. In our equation, if we apply $\lfloor . \rfloor$ to y_{ij}terms and knowing that $x_j s$ are *nonnegative*, we can write:

$$\sum_{j \notin B} \lfloor y_{ij} \rfloor x_j \leq \sum_{j \notin B} y_{ij} x_j .$$

Substituting to (1) we get:

$$x_{B(i)} + \sum_{j \notin B} \lfloor y_{ij} \rfloor x_j \leq y_{i0}\,, \qquad i = 0, 1, \dots, m \qquad (2)$$

Since we are seeking for integer solution *x*, the l.h.s. of (2) is now integer. Therefore, we can lower the r.h.s. to its largest integer without excluding any feasible integer
solution:

$$x_{B(i)} + \sum_{j \notin B} \lfloor y_{ij} \rfloor x_j \leq \lfloor y_{i0} \rfloor\,, \qquad i = 0, 1, \dots, m \qquad (3)$$

Subtract (3) from (1):

$$\sum_{j \notin B} (y_{ij} - \lfloor y_{ij} \rfloor)\, x_j \geq y_{i0} - \lfloor y_{i0} \rfloor\,, \qquad i = 0, 1, \dots, m \qquad (4)$$

Put $f_{ij} = y_{ij} - \lfloor y_{ij} \rfloor$, $i = 0, 1, \dots, m$. Where f_{ij} is the fractional part of y_{ij} which satisfies $0 \leq f_{ij} < 1$. Inequality (4) above can be written as:

$$\sum_{j \notin B} f_{ij} x_j \geq f_{i0}, \quad i = 0,1, \dots, m.$$

Which is the **Gomory's Fractional Cut** corresponding to row *i*.

How does it work?
Gomory's fractional cut generates a valid inequality for feasible integer set which does *not* exclude any integer solution. This statement follows from the fact that the cut is implied by the integrality requirements.
Now the question is, does it exclude the current (fractional) optimal solution?

Consider the **Gomory's Cut** again (for row *i*):

$$\sum_{j \notin B} f_{ij} x_j \geq f_{i0} \iff -\sum_{j \notin B} f_{ij} x_j \leq -f_{i0}$$

Introducing slack variable $s \geq 0$, above inequality results into:

$$-\sum_{j \notin B} f_{ij} x_j + s = -f_{i0} \qquad (5)$$

Since the current optimal solution $x_{B(i)}$ is fractional (otherwise we do not need to generate a cut corresponding to this row), we have $f_{i0} > 0$ (strictly positive). Rewrite (5):

$$s = -f_{i0} + \sum_{j \notin B} f_{ij} x_j = -f_{i0} + 0 < 0,$$

As x_j are non-basic variables and thus 0 at optimality. Above inequality yields $s < 0$ which violates the definition of a slack variable $s \geq 0$. This means *the current fractional optimal solution is made infeasible by adding the cut*, therefore the Gomory's cut as stated does remove part of the polyhedron for the LP relaxation.

We know how to generate a cut, now let's describe an algorithm that uses these cuts to solve IP problems. To start, two requirements must be satisfied (by the original IP) for this method to be applicable:

• All variables must be integer;
• All data (coefficients) must be integer.

As we only work on problems having rational parameters, as an initial step we can multiply each inequality by the least common multiple of its denominators to convert all terms into integers.

For example, if a constraint is given as $\frac{1}{2}x_1 + \frac{1}{3}x_2 \leq 4$, before starting the simplex algorithm we need to rewrite it into:

$$3x_1 + 4x_2 \leq 24$$

Procedure: Cutting Plane Algorithm

Step 0: Initialization: convert all coefficients of the IP into integers.
Step 1: Solve the **LP relaxation** of the IP by simplex algorithm.
Step 2: If the optimal solution is integer, ***stop***. Otherwise go to Step 3.
Step 3: Conduct cutting plane:
Step 3.1: Choose a source row from the simplex tableau corresponding to a basic variable with a fractional value nearest to 1/2.
Step 3.2: Generate a **Gomory's Cut**.
Step 3.3: Append the resulting constraint to the final tableau after adding a slack variable.
Step 3.4: Apply the **dual simplex algorithm** to the resulting tableau, since the current optimum is made primal infeasible by the cut.
Step 3.5: At optimality go to Step 2.

Example
Solve the following IP by Gomory's cutting plane algorithm and draw the cut generated on a graph.

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ s.t. \quad & \frac{1}{4}x_1 + \frac{1}{12}x_2 \leq 1 \\ & x_1 + x_2 \leq 5 \\ x_1, x_2 \geq 0 \ \ & and\ integer \end{aligned}$$

Solution:
Multiply constraint 2 by 12

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ s.t. \quad & 3x_1 + x_2 \leq 12 \\ & x_1 + x_2 \leq 5 \\ x_1, x_2 \geq 0 \ \ & and\ integer \end{aligned}$$

	z	x_1	x_2	s_1	s_2	s_3	r.h.s
	1	-5	-2	0	0		0
←	s_1	3	1	1	0		12
	s_2	1	1	0	1		5
	1	0	-1/3	5/3	0		20
	x_1	1	1/3	1/3	0		4
←	s_2	0	2/3	-1/3	1		1
		BV	BV	NBV	NBV		
	1	0	0	3/2	1/2	0	41/2
	x_1	1	0	1/2	-1/2	0	7/2
	x_2	0	1	-1/2	3/2	0	3/2
←	s_3	0	0	-1/2	-1/2	1	-1/2

** opt: (7/2, 3/2), z=20.5 not integral

(infeasible) Gomory's cut

Cut: $\left(\frac{1}{2} - \left\lfloor\frac{1}{2}\right\rfloor\right) s_1 + \left(-\frac{1}{2} - \left\lfloor-\frac{1}{2}\right\rfloor\right) s_2 \geq \left(\frac{7}{2} - \left\lfloor\frac{7}{2}\right\rfloor\right)$

$\left(\frac{1}{2} - 0\right) s_1 + \left(-\frac{1}{2} + 1\right) s_2 \geq (3.5 - 3)$ (Gomory's fractional cut)

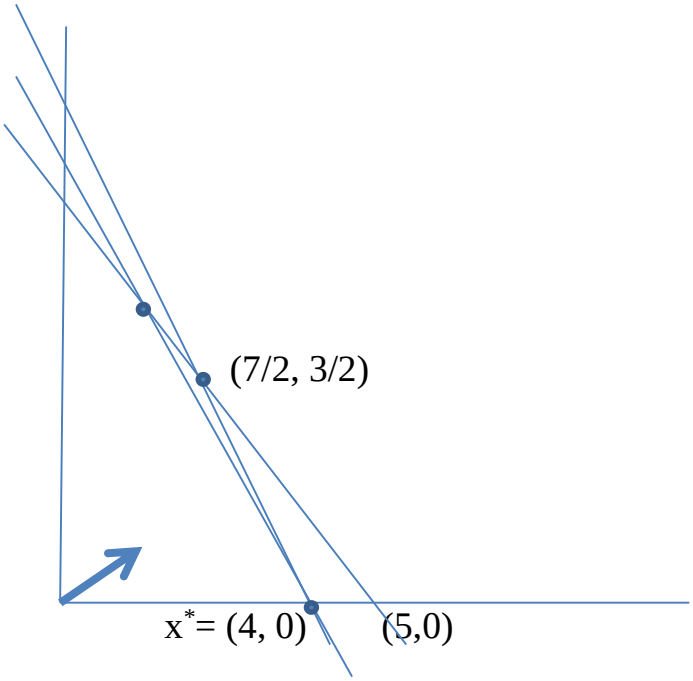
$\frac{1}{2} s_1 + \frac{1}{2} s_2 \geq \frac{1}{2}$ * multiply by -1

$-\frac{1}{2} s_1 - \frac{1}{2} s_2 \leq -\frac{1}{2}$

$\frac{-1}{2} s_1 - \frac{1}{2} s_2 + s_3 = \frac{-1}{2}$

z	x_1	x_2	s_1	s_2	s_3	r.h.s
1	0	0	1	0	1	20
x_1	1	0	1	0	-1	4
x_2	0	1	-2	0	3	0
s_2	0	0	1	1	-2	1

Opt: $x_1^* = 4,\ x_1^* = 0,\ z = 20$ stop



The cut we found by Gomory $\frac{1}{2}s_1 + \frac{1}{2}s_2 \geq \frac{1}{2}$ we can equivalently write it in terms of x_1 and x_2

$$s_1 = 12 - 3x_1 - x_2$$
$$s_2 = 5 - x_1 - x_2$$
$$\frac{1}{2}(12 - 3x_1 - x_2) + \frac{1}{2}(5 - x_1 - x_2) \geq \frac{1}{2} \quad *1/2$$
$$12 - 3x_1 - x_2 + 5 - x_1 - x_2 \geq 1$$
$$-4x_1 - 2x_2 \geq -16$$

Divide by 2

$$2x_1 + x_2 \leq 8$$

Remark

For mixed-integer programs there is a specialised Gomory's cut, but the finite convergence may not be guaranteed

H.W:

Solve the following IP by Gomory's cutting plane algorithm and draw the cut generated on a graph.

$$\max \quad z = 14x_1 + 18x_2$$
$$s.t. \quad -x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 35$$
$$x_1, x_2 \geq 0 \ \textit{and integers}$$

After introducing slack variables

Max $Z = 14x_1 + 18x_2 + 0S_1 + 0S_2$

subject to

$$-x_1 + 3x_2 + S_1 = 6$$
$$7x_1 + x_2 + S_2 = 35$$

and $x_1, x_2, S_1, S_2 \geq 0$

Iteration-1		C_j	14	18	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio $\frac{X_B}{x_2}$
S_1	0	6	-1	(3)	1	0	$\frac{6}{3} = 2 \rightarrow$
S_2	0	35	7	1	0	1	$\frac{35}{1} = 35$
$Z = 0$		Z_j	0	0	0	0	
		$Z_j - C_j$	-14	-18 ↑	0	0	

Iteration-2		C_j	14	18	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio $\frac{X_B}{x_1}$
x_2	18	2	$-\frac{1}{3}$	1	$\frac{1}{3}$	0	---
S_2	0	33	$\left(\frac{22}{3}\right)$	0	$-\frac{1}{3}$	1	$\frac{33}{\frac{22}{3}} = \frac{9}{2} = 4.5 \rightarrow$
$Z = 36$		Z_j	-6	18	6	0	
		$Z_j - C_j$	-20 ↑	0	6	0	

Iteration-3		C_j	14	18	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio
x_2	18	$\frac{7}{2}$	0	1	$\frac{7}{22}$	$\frac{1}{22}$	
x_1	14	$\frac{9}{2}$	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	
$Z = 126$		Z_j	14	18	$\frac{56}{11}$	$\frac{30}{11}$	
		$Z_j - C_j$	0	0	$\frac{56}{11}$	$\frac{30}{11}$	

Since all $Z_j - C_j \geq 0$

Hence, optimal solution is arrived with value of variables as :

$x_1 = \frac{9}{2}, x_2 = \frac{7}{2}$

Max $Z = 126$